

Equations of Motion of Spinning Relativistic Particles

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Abstract

The motion of spinning relativistic particles in external electromagnetic and gravitational fields is considered. Covariant equations for this motion are demonstrated to possess pathological solutions, when treated nonperturbatively in spin. A self-consistent approach to the problem is formulated, based on the noncovariant description of spin and on the usual, “naïve” definition of the coordinate of a relativistic particle. A simple description of the gravitational interaction of first order in spin, is pointed out for a relativistic particle. The approach developed allows one to consider effects of higher order in spin. Explicit expression for the second-order Hamiltonian is presented. We discuss the gravimagnetic moment, which is a special spin effect in general relativity.

1 Introduction

The problem of the motion of a particle with internal angular momentum (spin) in an external field consists of two parts: the description of the spin precession and accounting for the spin influence on the trajectory of motion. To lowest nonvanishing order in c^{-2} the complete solution for the case of an external electromagnetic field was given more than 70 years ago [1]. The gyroscope precession in a centrally symmetric gravitational field had been considered to the same approximation even earlier [2]. The fully relativistic problem of the spin precession in an external electromagnetic field was also solved more than 70 years ago [3], and then in a more convenient formalism, using the covariant vector of spin, in [4].

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The situation is different with the second part of the problem, which refers to the spin influence on the trajectory. Covariant equations of motion for a relativistic spinning particle in an electromagnetic field were written in the same paper [3], and for the case of a gravitational field in [5]. Then these equations were discussed repeatedly from various points of view in numerous papers. The problem of the influence of the spin on the trajectory of a particle in external fields is not only of purely theoretical interest. It is related to the description of the motion of relativistic particles in accelerators [6] (see also review [7]). There are also macroscopic objects for which internal rotation certainly influences their trajectories. We mean the motion of Kerr black holes in external gravitational fields.

Here we will elucidate serious shortcomings of the covariant description of the spin influence on the trajectory, and formulate a self-consistent non-covariant approach to the problem. Most of the results presented below were worked out in collaboration with A. Pomeransky and R. Sen'kov. Our papers [8-10] contain more details, as well as more complete list of references.

2 What is wrong with covariant equations of motion?

A covariant correction f^μ to the Lorentz force $eF^{\mu\nu}u_\nu$ should be linear in the tensor of spin $S_{\mu\nu}$ and in the gradient of the tensor of electromagnetic field $F_{\mu\nu,\lambda}$, it may depend also on the 4-velocity u^μ . Since $u^\mu u_\mu = 1$, this correction must satisfy the condition $u_\mu f^\mu = 0$. From the mentioned tensors one can construct only two independent structures meeting the last condition. The first, $\eta^{\mu\kappa}F_{\nu\lambda,\kappa}S^{\nu\lambda} - F_{\lambda\nu,\kappa}u^\kappa S^{\lambda\nu}u^\mu$, reduces in the c^{-2} approximation to

$$2\mathbf{s}(\mathbf{B}_{,m} - [\mathbf{v} \times \mathbf{E}_{,m}]),$$

and the second, $u^\lambda F_{\lambda\nu,\kappa}u^\kappa S^{\nu\mu}$, reduces to

$$\frac{d}{dt} [\mathbf{s} \times \mathbf{E}].$$

Here \mathbf{B} and \mathbf{E} are external magnetic and electric fields; e , m , \mathbf{s} , and \mathbf{p} are the particle charge, mass, spin, and momentum, respectively; g is its gyromagnetic ratio; a comma with a subscript denotes a partial derivative.

Obviously, no linear combination of these two structures can reproduce the correct result for the spin-dependent force,

$$f_m = \frac{eg}{2m} \mathbf{s} \mathbf{B}_{,m} + \frac{e(g-1)}{2m} \left(\frac{d}{dt} [\mathbf{E} \times \mathbf{s}]_m - \mathbf{s} [\mathbf{v} \times \mathbf{E}_{,m}] \right),$$

which follows in the c^{-2} approximation from the well-known noncovariant Hamiltonian (see, e.g., [11])

$$H_{e1} = \boldsymbol{\Omega} \mathbf{s} = \frac{e}{2m} \mathbf{s} \left\{ (g-2) \left[\mathbf{B} - \frac{\gamma}{\gamma+1} \mathbf{v}(\mathbf{v} \mathbf{B}) - \mathbf{v} \times \mathbf{E} \right] + 2 \left[\frac{1}{\gamma} \mathbf{B} - \frac{1}{\gamma+1} \mathbf{v} \times \mathbf{E} \right] \right\}, \quad \gamma = \frac{1}{\sqrt{1-v^2}}. \quad (1)$$

Let us emphasize, that though being noncovariant, this Hamiltonian is valid for arbitrary velocities.

Here the covariant formalism can be reconciled with the correct results if the coordinate \mathbf{x} entering the covariant equation is related to the usual one \mathbf{r} as follows in the c^{-2} approximation:

$$\mathbf{x} = \mathbf{r} + \frac{1}{2m} \mathbf{s} \times \mathbf{v}. \quad (2)$$

The generalization of this substitution to the case of arbitrary velocities is [7]:

$$\mathbf{x} = \mathbf{r} + \frac{\gamma}{m(\gamma+1)} \mathbf{s} \times \mathbf{v}. \quad (3)$$

However, after this velocity-dependent substitution, the equations of motion depend on the third time derivative of coordinate. This is harmless by itself as long as the corresponding term is treated as perturbation. But beyond the perturbation theory these equations possess fictitious unphysical solutions. Let us demonstrate it explicitly with the simplest possible example of free motion. Free covariant equations are (see, e.g., [12]):

$$\frac{d}{d\tau} (mu_\mu - S_{\mu\nu} \dot{u}_\nu) = 0, \quad \dot{S}_{\mu\nu} + (u_\nu S_{\mu\lambda} - u_\mu S_{\nu\lambda}) \dot{u}_\lambda = 0 \quad (4)$$

Integrating the first of them, we obtain $mu_\mu - S_{\mu\nu} \dot{u}_\nu = c_\mu$, where c_μ is a constant 4-vector. Then the second equation reduces to $\dot{S}_{\mu\nu} = c_\mu u_\nu - c_\nu u_\mu$. Of course, the physical, free solution $u_\mu = c_\mu/m$, $\dot{S}_{\mu\nu} = 0$ does exist.

But equations (4) have one more family of solutions. To investigate and describe them, it is convenient to introduce the spin 4-vector S_μ related to the tensor $S_{\mu\nu}$ as follows: $S_{\mu\nu} = \varepsilon_{\mu\nu\rho\tau} S_\rho u_\tau$. The solutions we are looking for, can be chosen in such a way that $S_\mu = (0, 0, 0, s)$, $u_3 = 0$, $c_3 = 0$. Then the second of the equations reduces to

$$s\varepsilon_{\mu\nu\rho}\dot{u}_\rho = c_\mu u_\nu - c_\nu u_\mu, \quad \text{or} \quad \dot{u}_\rho = \frac{1}{s}\varepsilon_{\rho\mu\nu}c_\mu u_\nu \quad (5)$$

(from now on, in formulae related to this solution indices run through 0,1,2). Equation $mu_\mu - S_{\mu\nu}\dot{u}_\nu = c_\mu$ is satisfied identically with (5).

If the constant vector c_μ is time-like, we can choose the reference frame in such a way that $c_\mu = (m/u_0, 0, 0)$ with $u_0 = \text{const}$ (recall the condition $c_\mu u_\mu = m$). The energy is conserved, and Eq. (5) describes obviously the precession of the particle velocity with respect to its spin with the frequency $\omega = m/u_0 s$ (in the proper time τ).

Another option is a space-like c_μ . Then choosing $c_\mu = (0, 0, -m/u_2)$ with $u_2 = \text{const}$, we obtain self-acceleration along the axis 1: $u_0 \sim \cosh g\tau$, $u_1 \sim \sinh g\tau$, $g = m/u_2 s$. One cannot but recall here the self-acceleration of radiating electron in classical electrodynamics.

Obviously, equations with pathological solutions cannot have a fundamental meaning.

At last, let us demonstrate that it is the naïve, common coordinate \mathbf{r} , rather than \mathbf{x} , which should be considered as the true coordinate of a relativistic spinning particle. Since relations (2), (3) are valid for a free particle as well, the problem can be elucidated with a simple example of a free particle with spin 1/2. Here, instead of the Dirac representation with the Hamiltonian of the standard form

$$H_D = \boldsymbol{\alpha}\mathbf{p} + \beta m,$$

it is convenient to use the Foldy-Wouthuysen (FW) representation. In it the Hamiltonian is

$$H_{FW} = \beta\varepsilon_{\mathbf{p}}, \quad \varepsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2},$$

and the 4-component wave functions ψ_\pm of the states of positive and negative energies reduce in fact to the 2-component spinors ϕ_\pm :

$$\psi_+ = \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ \phi_- \end{pmatrix}.$$

Obviously, in this representation the operator of coordinate $\hat{\mathbf{r}}$ defined by the usual relation

$$\hat{\mathbf{r}}\psi(\mathbf{r}) = \mathbf{r}\psi(\mathbf{r}), \quad (6)$$

is just \mathbf{r} .

The transition from the exact Dirac equation in an external field to its approximate form containing only the first-order correction in c^{-2} , is performed just by means of the FW transformation. Thus, in the resulting c^{-2} Hamiltonian the coordinate of a spinning electron is the same \mathbf{r} as in the completely nonrelativistic case. Nobody makes substitution (2) in the Coulomb potential when treating the spin-orbit interaction in the hydrogen atom.

One more limiting case, which is of a special interest to us, is a classical spinning particle. Such a particle is in fact a well-localized wave packet constructed from positive-energy states, i.e., it is naturally described in the FW representation. Therefore, it is just \mathbf{r} which it is natural to consider as the coordinate of a relativistic spinning particle.

A certain subtlety here is that in the Dirac representation the operator $\hat{\mathbf{r}}$ is nondiagonal. However, the operator equations of motion certainly have the same form both in the Dirac and Foldy-Wouthuysen representations. Correspondingly, the semiclassical approximation to both is the same. In particular, the time derivatives in the left-hand side of classical equations of motion are taken of the same coordinate \mathbf{r} , which serves as an argument of the fields in the right-hand side of these equations.

3 Effects of higher order in spin. The idea of general formalism

The effects linear in spin for the motion of a spinning particle in an electromagnetic field are described by the noncovariant Hamiltonian (1). The noncovariant Hamiltonian for first-order spin effects in a gravitational field can be also obtained from (1) (see [9, 10]) by putting $g = 2$ and substituting

$$\frac{e}{m}B_i \longrightarrow -\frac{1}{2}\varepsilon_{ikl}\gamma_{klc}u^c; \quad \frac{e}{m}E_i \longrightarrow \gamma_{0ic}u^c, \quad \gamma = \frac{1}{\sqrt{1-v^2}} \longrightarrow u_w^0. \quad (7)$$

Here $\gamma_{abc} = e_{a\mu;\nu}e_b^\mu e_c^\nu = -\gamma_{bac}$ are the Ricci rotation coefficients. A subscript w is attached to the quantity u_w^0 to emphasize that u_w^0 is a world, but not a

tetrad, component of 4-velocity. All other indices in expression (7) are tetrad ones, $a, b, c = 0, 1, 2, 3$; $i, k, l = 1, 2, 3$.

However, at least in the motion of rotating black holes (and possibly in some subtle spin effects for polarized nuclei of high spin in storage rings) the interaction of second order in spin may manifest itself. Anyway, going beyond the linear approximation in spin is of a certain theoretical interest. To study this general problem, a more sophisticated approach is needed. It is based on the following physically obvious argument: as long as we do not consider excitations of internal degrees of freedom of a body moving in an external field, this body (even if it is a macroscopic one!) can be treated as an elementary particle with spin. Thus, the Hamiltonian of the spin interaction with an external field can be derived from the elastic scattering amplitude of a particle with spin s by external field. In this way we can describe the interaction of a relativistic particle to arbitrary order in the spin.

The details of the approach can be found in [9, 10]. Here we present only the expression for the second-order (in spin) electromagnetic interaction:

$$\begin{aligned}
H_{e2} = & -\frac{Q}{2s(2s-1)} \left[(\mathbf{s}\nabla) - \frac{\gamma}{\gamma+1} (\mathbf{v}\mathbf{s})(\mathbf{v}\nabla) \right] \\
& \times \left[(\mathbf{s}\mathbf{E}) - \frac{\gamma}{\gamma+1} (\mathbf{s}\mathbf{v})(\mathbf{v}\mathbf{E}) + (\mathbf{s}[\mathbf{v} \times \mathbf{B}]) \right] \\
& + \frac{e}{2m^2} \frac{\gamma}{\gamma+1} (\mathbf{s}[\mathbf{v} \times \nabla]) \left[\left(g - 1 + \frac{1}{\gamma} \right) (\mathbf{s}\mathbf{B}) \right. \\
& \left. - (g-1) \frac{\gamma}{\gamma+1} (\mathbf{s}\mathbf{v})(\mathbf{v}\mathbf{B}) - \left(g - \frac{\gamma}{\gamma+1} \right) (\mathbf{s}[\mathbf{v} \times \mathbf{E}]) \right]. \tag{8}
\end{aligned}$$

Here the particle quadrupole moment Q is defined as usual: $Q = Q_{zz}|_{s_z=s}$.

Of great interest is the asymptotic behaviour of the interaction (8) at $\gamma \rightarrow \infty$. Though both Q -dependent and Q -independent parts of the interaction (8) grow up when taken separately, there is a singled out value of the quadrupole moment for which this interaction as a whole falls down with energy.

It is well-known (and follows immediately from (1)) that there is a special value of the g -factor, $g = 2$, for which the electromagnetic interaction linear in spin decreases with increasing energy. Thus, the choice $g = 2$ for the bare magnetic moment is a necessary (but insufficient!) condition of unitarity and renormalizability in quantum electrodynamics. It holds not only for

the electron, but also for the charged vector boson in the renormalizable electroweak theory.

The same situation takes place with the second-order spin interaction in electrodynamics. There is a special value of the quadrupole moment Q at which this interaction as well decreases with increasing energy. If we also assume $g = 2$, this value is

$$Q = -s(2s - 1) \frac{e}{m^2}. \quad (9)$$

Again, (9) is a necessary condition of unitarity and renormalizability. And indeed, this is the value of the quadrupole moment of the charged vector boson in the renormalizable electroweak theory. For it $g = 2$, $s = 1$, $Q = -e/m^2$.

4 Gravimagnetic moment. Multipoles of black holes

For a binary star effects of second-order in spin are of the same order of magnitude as the spin-spin interaction. Second-order spin effects in the equations of motion become substantial if at least one component of a binary is close to an extreme black hole.

The equations of motion in an external gravitational field to any order in spin can be obtained within our general approach as well [9]. However, in this brief contribution we confine ourselves to an instructive short-cut which allows one to derive without lengthy calculations the so-called gravimagnetic interaction [13], a gravitational analogue of the Q -dependent terms in (8).

In fact, the analogy between first-order spin interactions in electrodynamics and gravity is incomplete. While the electromagnetic interaction depends on the field strength, which is gauge-invariant, the gravitational one depends not on the Riemann tensor, which is generally covariant, but on the Ricci rotation coefficients, which are not. This is only natural: in a flat space, spin which is at rest in an inertial frame, precesses in a rotating frame.

In this respect, the second-order spin interaction discussed below, the gravimagnetic one, which depends on the Riemann tensor, is the gravitational analogue of the first-order spin interactions in electrodynamics. Our starting point is the observation that the canonical momentum p_μ enters a relativistic wave equation for a particle in external electromagnetic and gravitational

fields through the combination $\Pi_\mu = p_\mu - eA_\mu - (1/2)\Sigma^{ab}\gamma_{ab\mu}$. Here Σ^{ab} are the generators of the Lorentz group; $\gamma_{ab\mu} = e_\mu^c\gamma_{abc}$. The commutation relation

$$[\Pi_\mu, \Pi_\nu] = -ieF_{\mu\nu} + \frac{i}{2}\Sigma^{ab}R_{ab\mu\nu} \quad (10)$$

demonstrates the remarkable correspondence

$$eF_{\mu\nu} \longleftrightarrow -\frac{1}{2}\Sigma^{ab}R_{ab\mu\nu}. \quad (11)$$

The squared form of the Dirac equation in an external electromagnetic field prompts that for an arbitrary spin s the Hamiltonian $e/(2m)\Sigma^{ab}F_{ab}$ describes the magnetic moment interaction for $g = 2$. Clearly, for an arbitrary g -factor this covariant magnetic moment interaction is

$$\mathcal{H}_{e1} = \frac{eg}{4m}F_{ab}\Sigma^{ab}. \quad (12)$$

This is in fact a covariant form of g -dependent terms in the Hamiltonian (1). It is natural to define in analogy with the magnetic moment

$$\frac{eg}{2m}\Sigma^{ab},$$

the gravimagnetic moment

$$-\frac{\kappa}{2m}\Sigma^{ab}\Sigma^{cd}.$$

Now, the correspondence (11) prompts the following gravitational analogue of the Lagrangian (12):

$$\mathcal{H}_{gm} = -\frac{\kappa}{8m}\Sigma^{ab}\Sigma^{cd}R_{abcd}. \quad (13)$$

This is what we call the gravimagnetic interaction. Let us note that in the classical limit $\Sigma^{ab} \rightarrow S^{ab} = \varepsilon^{abcd}S_c u_d$.

The gravimagnetic ratio κ , like the gyromagnetic ratio g in electrodynamics, may have in general any value. Still, it is natural that in gravity the value $\kappa = 1$ is as singled out as $g = 2$ in electrodynamics. Indeed, the analysis of the complete noncovariant Hamiltonian for the gravitational interaction of second order in spin, including of course κ -independent terms which correspond to the Q -independent terms in (8), demonstrate that just

for $\kappa = 1$ this total interaction asymptotically tends to zero with increasing energy. However, the gravitational interaction for any spin is not renormalizable even at $\kappa = 1$.

In any case, for $g = 2$ and $\kappa = 1$ the equations of motion have the simplest form. Moreover, just this value of the gravimagnetic ratio, $\kappa = 1$, follows from the wave equations for the graviton and spin-3/2 particle in an external gravitational field.

Wave equations for particles of arbitrary spins in an external gravitational field were previously considered in [14]. The equation for integer spins proposed in [14] corresponds also to the gravimagnetic ratio $\kappa = 1$. However, the value of κ prescribed in [14] for half-integer spins is different. Even in the classical limit $s \rightarrow \infty$, this value does not tend to unity. This obviously does not comply with the correspondence principle: at least in this classical limit there should be no difference between integer and half-integer spins.

Let us come back from elementary particles to macroscopic bodies. For a classical object the values of both parameters g and κ depend in general on the various properties of the body. However, for black holes the situation is different. The gyromagnetic ratio of a charged rotating black hole is universal (and equal to that of the electron!): $g = 2$ [15]. As universal is the gravimagnetic ratio of the Kerr black hole: $\kappa = 1$. Moreover, the electric quadrupole moment of a charged Kerr hole also equals $Q = -2es^2/m^2$, the value, at which the interaction quadratic in spin decreases with energy (this is the obvious limit of the general formula (9) at $s \rightarrow \infty$). Other, higher multipoles of a charged Kerr hole, both electromagnetic and gravitational, as well possess just those values which guarantee that the interaction of any order in spin (but linear in an external field) asymptotically decreases with increasing energy [16].

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